Revealing intermittency in experimental data with steep power spectra

E. Falcon[1], S. G. Roux[2] and B. Audit[3]

[1] *Laboratoire Matière et Systèmes Complexes (MSC), Université Paris Diderot, CNRS – UMR 7057*
10 rue A. Domon & L. Duquet, 75013 Paris, France, EU
[2] *Université de Lyon, Laboratoire de Physique, CNRS, Ecole Normale Supérieure de Lyon, 69007 Lyon, France, EU*
[3] *Université de Lyon, Laboratoire Joliot-Curie, CNRS, Ecole Normale Supérieure de Lyon, 69007 Lyon, France, EU*

PACS 05.40.-a – Fluctuation phenomena, random processes
PACS 47.27.-i – Turbulent flows
PACS 47.35.-i – Hydrodynamic waves

Abstract. - The statistics of signal increments are commonly used in order to test for possible intermittent properties in experimental or synthetic data. However, for signals with steep power spectra [i.e., $E(\omega) \sim \omega^{-n}$ with $n \geq 3$], the increments are poorly informative and the classical phenomenological relationship between the scaling exponents of the second-order structure function and of the power spectrum does not hold. We show that in these conditions the relevant quantities to compute are the second or higher degree differences of the signal. Using this statistical framework to analyze a synthetic signal and experimental data of wave turbulence on a fluid surface, we accurately characterize intermittency of these data with steep power spectra. The general application of this methodology to study intermittency of experimental signals with steep power spectra is discussed.

Introduction. – Since the prediction of Kolmogorov in 1941 [1], it is well known that the spatial power spectrum $E(k)$ of a fluid particle velocity v in a turbulent flow is a power-law of the wave number k as $k^{-5/3}$. The $-5/3$ exponent of the spatial power spectrum is related to the second-order moment of velocity increments $S_2(r) \equiv \langle [v(l+r) - v(l)]^2 \rangle \sim r^{2/3}$, l and r being a position and a spatial separation [2]. The phenomenological relationship between both exponents comes from Fourier transform properties. It can be generalized to any stationary random processes: if the power spectrum of the process is $E(k) \sim k^{-n}$, then $S_2(r) \sim r^{\zeta_2}$ with $\zeta_2 = n - 1$ [2]. This property allows to perform measurements in the real space to reach the power-law exponent of the power spectrum. The statistics of velocity increments are also crucial to characterize the intermittent nature of the velocity field using the scaling properties of structure functions: $S_p(r) \equiv \langle [v(l+r) - v(l)]^p \rangle \sim r^{\zeta_p}$ (p positive integer) [3]. A non-linear dependence of ζ_p versus p is the hallmark of intermittency.

Steep power-law spectra ($\sim \omega^{-n}$ or $\sim k^{-n}$ with n close or larger than 3) of a process are observed in various situations: magnetohydrodynamics turbulence [4], atmospherics turbulence [5], gravity [6] or capillary [7] wave turbulence on a fluid surface, and direct cascade of two-dimensional fluid turbulence [8, 9]. Whatever the corresponding signal measured in space or in time (*e.g.* fluid velocity or vorticity, surface-wave height, magnetic field, wind), such steep spectra mean that the measured signal is at least once continuously differentiable [2]. The signal differences or increments are thus poorly informative since they are dominated by the differentiable component of the signal. For instance, some numerical simulations of the power-law scaling of the energy spectrum in two-dimensional turbulence exhibited apparent contradictions with its reconstruction from spatial correlation measurement, i.e. $\zeta_2 \neq n - 1$ (see Ref. [10]). Babiano *et al.* have systematically reconsidered the theoretical relations between second-order structure functions and energy spectra instead of phenomenological or dimensional arguments [10]. They showed that the apparent contradictions come from the fact that the relation $\zeta_2 = n - 1$ does not hold for steep power-law spectra. Indeed, due to the differentiable component, the exponent of the second-order structure function is independent of the spectrum slope as soon as this one is steeper than -3; that is $\zeta_2 = 2$

whatever $n \geq 3$ [10]. This latter property has been noted elsewhere without derivation [2, 11, 12]. The indirect conclusions drawn from the structure function analysis to reach the exponent of the power-law spectrum (using the relation $\zeta_2 = n - 1$) are thus misleading for $n \geq 3$, but suprinsingly are still used for some experimental signals with steep power spectra [13].

In this letter, we emphasize that the increments are not the relevant quantities in order to statistically characterize the fluctuations of a steep power spectrum signal and, in particular, to probe for its possible intermittent nature. We show analytically that, depending on the spectral steepness, it is necessary to adapt the degree of the difference statistics used to analyze the signal. Given an adapted degree, we provide a general relationship between the exponents of the second-order structure function and of the power spectrum whatever the steepness of the spectrum. Finally, applying this approach to a synthetic signal, and to an experimental signal of wave turbulence on a fluid surface allows us to accurately characterize intermittency of these data with steep power spectra. Note that a general framework for the study of intermittency of a signal with arbitrary degree of regularity has been previously proposed using more complex estimators based on the continuous wavelet transform [14] or on inverse statistics [15]. Here, we propose a practical approach and provide simple rules that should be easily applicable on the workbench to study intermittency of experimental signals with steep spectra.

Scaling properties of irregular signals. – Let us first recall the pioneering work of Parisi and Frisch [3] introduced for the description of the irregular nature of longitudinal velocity data in fully developed turbulence. They locally described the fluctuations of an erratic signal, $\eta(t)$, by means of the singularity exponents $h(t_0)$ which caracterizes the power-law behaviour of the finite differences (or increments) of η over a time lag τ at a time t_0

$$\delta_\tau\eta(t_0) \equiv \eta(t_0+\tau) - \eta(t_0) \underset{\tau\mapsto 0^+}{\propto} \tau^{h(t_0)} \ . \qquad (1)$$

Since the characterization of non-local singularities cannot be achieved in a purely local manner, Parisi and Frisch introduced the p-order structure functions

$$S_p(\tau) \equiv \langle |\delta_\tau\eta(t)|^p \rangle \ , \qquad (2)$$

where $\langle\cdot\rangle$ represents an average over time t and from which they defined the spectra of global exponents ζ_p such as

$$S_p(\tau) \propto \tau^{\zeta_p} \ . \qquad (3)$$

Note that it is useful to consider the moment of order p of $|\delta_\tau\eta(t)|$ rather than $\delta_\tau\eta(t)$ as used in Ref. [3], so that Eqs. (2) and (3) are defined for all p positive real. In the presence of homogeneous fluctuations, i.e. $h(t_0) = H$ whatever t_0, it is straightforward that ζ_p is a linear function of p with $\zeta_p = pH$. Reciprocally, a non-linear dependence of ζ_p with p is the signature of non-homogeneous fluctuations whose singularity exponents vary with time, i.e. the hallmark of intermittency.

Scaling exponents of two point statistics. – We now resume the relationships between the power-law exponents of the second-order structure functions, the correlation function, and the power spectrum. Let us consider a stochastics process $\eta(t)$ which is not necessarily stationary (see below). We assume that finite differences of η over a time lag τ, $\delta_\tau\eta(t) \equiv \eta(t+\tau) - \eta(t)$, form a stationary process of zero mean and that $\eta(0) = 0$. Let us look at the correlation between two short intervals (of size Δ) separated by a lag θ, that is the quantity $\delta_\Delta\eta(t+\theta)\delta_\Delta\eta(t)$. We choose Δ equals to the sampling time of $\eta(t)$, that is the duration between two successive measurement points. For $p = 2$, the scaling of Eq. (3) implies a power-law decay of the correlation function C of the increments $\delta_\Delta\eta$ at large lags θ as [16]

$$C(\theta) \equiv \langle \delta_\Delta\eta(t+\theta)\delta_\Delta\eta(t)\rangle \underset{|\theta|\mapsto\infty}{\propto} |\theta|^{-\kappa} \text{ , with } \kappa = 2 - \zeta_2 \ . \qquad (4)$$

This scaling behaviour for large lags coincides to a low-frequency power-law behaviour of the power spectrum $\widehat{C}$ of the increments $\delta_\Delta\eta$, defined as the Fourier transform (FT) of the correlation function [16]

$$\widehat{C}(\omega) \equiv \int_{\mathbb{R}} C(\theta)\mathrm{e}^{i\omega\theta}\mathrm{d}\theta \underset{|\omega|\mapsto 0}{\propto} |\omega|^{-\beta} \text{ , with } \beta = \zeta_2 - 1 \ , \qquad (5)$$

ω being the angular frequency. An estimator of the power spectrum $\widehat{C}(\omega)$ is obtained taking the square modulus of the FT of the increments $\delta_\Delta\eta(t)$ observed over a finite time range $[0, T]$,

$$\widehat{C}(\omega) \simeq E_{\delta_\Delta\eta}(\omega) \equiv \left| \frac{1}{T}\int_0^T \delta_\Delta\eta(t)\mathrm{e}^{i\omega t}\mathrm{d}t \right|^2 \ . \qquad (6)$$

In practice, a common habit is to compute the empirical power spectrum of η, denoted $E(\omega)$, rather than that of the increments, $E_{\delta_\Delta\eta}(\omega)$. Indeed, these spectra are related by $E_{\delta_\Delta\eta}(\omega) = 2\,[1-\cos(\omega)]\,E(\omega)$ leading to the empirical power spectrum scaling

$$E(\omega) \simeq \frac{\widehat{C}(\omega)}{2\,[1-\cos(\omega)]} \underset{|\omega|\mapsto 0}{\propto} |\omega|^{-n} \text{ , with } n = \beta + 2 \ . \qquad (7)$$

The scaling exponents of $S_2(\tau) \propto \tau^{\zeta_2}$, $C(\theta) \propto |\theta|^{-\kappa}$ and $\widehat{C}(\omega) \propto \omega^{-\beta}$ for the increments thus read respectively

$$\zeta_2 = n - 1, \ \ \kappa = 3 - n, \text{ and } \ \beta = n - 2, \qquad (8)$$

where $n < 3$ (see below) is the exponent of the power spectrum of $\eta(t)$ [see Eq. (7)].

It is fundamental to note that the above stationarity condition for the increments does not imply the stationarity of the signal, so that the correlation function of η might not be defined and, thus, that E may not be a power spectrum in the statistical sense (*i.e.* the FT of

Table 1: Relationships between the spectral exponent n, the structure function exponent and the differentiability of a signal $\eta(t)$ with a self-similar power spectrum. Reference indicates the existing derivation of the relationships. $S_2^{(1)}(\tau) \equiv \langle|\delta_\tau^{(1)}\eta|^2\rangle$, $S_2^{(2)}(\tau) \equiv \langle|\delta_\tau^{(2)}\eta|^2\rangle$ and $S_2^{(3)}(\tau) \equiv \langle|\delta_\tau^{(3)}\eta|^2\rangle$.

Power spectrum $E(\omega) \sim \omega^{-n}$	Differen-tiability	Difference statistics used to test intermittency	Second-order structure funct.
$n < 3$	0	$\delta_\tau^{(1)}\eta = \eta(t+\tau) - \eta(t)$	$S_2^{(1)} \sim \tau^{n-1}$ [10]
$n \geq 3$	≥ 1	-	$S_2^{(1)} \sim \tau^2$ [10]
$n < 5$	1	$\delta_\tau^{(2)}\eta = \eta(t+2\tau) - 2\eta(t+\tau) + \eta(t)$	$S_2^{(2)} \sim \tau^{n-1}$
$n \geq 5$	≥ 2	-	$S_2^{(2)} \sim \tau^4$
$n < 7$	2	$\delta_\tau^{(3)}\eta = \eta(t+3\tau) - 3\eta(t+2\tau) + 3\eta(t+\tau) - \eta(t)$	$S_2^{(3)} \sim \tau^{n-1}$
$n \geq 7$	≥ 3	-	$S_2^{(3)} \sim \tau^6$

a correlation function). Also, one should be careful that the estimation of the spectrum $E(\omega)$ for a non-stationary signal $\eta(t)$ may be significantly biased depending on the FT numerical algorithm used. In such condition, a more robust practice to estimate the spectral scaling exponent n consists in numerically computing the FT of the stationary signal $\delta_\Delta\eta(t)$, *i.e.*, $E_{\delta_\Delta\eta}(\omega)$, and to use Eq. (7) to convert back to the usual power spectrum.

Meaning of steep power spectra. – Here, we explain why the usual relationships of Eq. (8) no longer hold in the presence of steep power spectra ($n \geq 3$). In this case, Eq. (8) leads to $\kappa < 0$, suggesting that the correlation function of $\delta_\Delta\eta$ does not go to 0 but rather diverge at large lag values [see Eq. (4)]. Obviously, this is neither physically nor statistically acceptable. Indeed, for a non trivial stationary process $\delta_\tau\eta$ satisfying Eq. (3), the correlation function of Eq. (4) is only defined for $\kappa = 2-\zeta_2 > 0$, that is for $\zeta_2 < 2$. This implies that the spectrum of the increments is only defined for $\beta = \zeta_2 - 1 < 1$. Consequently, for a process with stationary increments, the scaling exponent of the empirical power spectrum must satisfies $n = \zeta_2 + 1 < 3$. Hence, for steep power spectra ($n \geq 3$), the basic assumption that increments form a stationary process is not verified so that the structure functions of Eq. (3) and the correlation function of Eq. (4) are not well defined. In practice, the classical phenomenological relation between $E(\omega) \sim \omega^{-n}$ and $S_2(\tau) \sim \tau^{n-1}$ is thus invalid for $n \geq 3$, so that the spectral slope can not be deduced from the measurement of the second-order structure function. This also means that the process $\eta(t)$ is at least once differentiable at times where $\eta(t+\tau) - \eta(t) \simeq \tau d\eta/dt$ at the first order in τ [2]. These local linear trends are responsible for the non-stationarity of signal increments and, in turns, bias the estimation of scaling exponents. Indeed, near these times, one has $|\delta_\Delta\eta(t)|^p \sim \tau^p$ that corresponds to $\zeta_p = p$ [using Eqs. (2) and (3)]. This means that the scaling of the exponent of the structure functions are independent of the spectral steepness. Thus, the increments of the signal do not appear as relevant quantities when looking for possible intermittency (i.e. a non-linear evolution of ζ_p with p) in signals with steep power spectra.

Using higher-degree difference statistics to recover stationarity. – As recalled above, the scaling exponent of the spectrum of the increments is decreased by two with respect to the one of the spectrum of the signal ($\beta = n-2$). Clearly, the repetition of the difference process allows recovering the power spectrum of a stationary process. For instance, for $3 \leq n < 5$, the second-degree difference of the signal $\delta_\Delta^{(2)}\eta(t) \equiv \delta_\Delta[\delta_\Delta\eta(t)] = \eta(t+2\Delta) - 2\eta(t+\Delta) + \eta(t)$ has a power spectrum with scaling exponent $\beta^{(2)} = n-4 < 1$, which is compatible with $\delta_\Delta^{(2)}\eta(t)$ being stationary. The second-degree differences thus remove the local linear trends in the signal η responsible for the saturation $\zeta_p = p$ when $n \geq 3$. Thus, when looking for possible intermittency in this case, one should use the structure functions of degree 2, that is $S_p^{(2)}(\tau) \equiv \langle|\delta_\tau^{(2)}\eta(t)|^p\rangle \propto |\tau|^{\zeta_p^{(2)}}$. We have indeed $\zeta_2^{(2)} = n-1$ for $n < 5$. For $n \geq 5$, one have $\zeta_2^{(2)} = 4$ due to local quadratic trends in the signal. Thus, for $5 \leq n < 7$, the power spectrum of third-degree differences $\delta_\Delta^{(3)}\eta(t) \equiv \delta_\Delta[\delta_\Delta^{(2)}\eta(t)] = \eta(t+3\Delta) - 3\eta(t+2\Delta) + 3\eta(t+\Delta) - \eta(t)$, is well defined and intermittency should be tested with $S_p^{(3)}(\tau) \equiv \langle|\delta_\tau^{(3)}\eta(t)|^p\rangle \propto |\tau|^{\zeta_p^{(3)}}$. These results are summarized in the Table 1. Note that, even though the slope of the power spectrum is reduced by computing higher-degree difference statistics, the upper bound of the spectral bandwidth, related to the finite dynamic resolution of the original signal measurement, is not bypassed.

Workbench recipe. – In practice, given a first estimate of the empirical power spectrum scaling exponent n, further statistical analysis should be performed using difference statistics of degree $d > d^*$ where d^* is the smallest integer such that $n - 2d^* < 1$. For instance, structure function of degree d, $S_p^{(d)}(\tau) \equiv \langle|\delta_\tau^{(d)}\eta(t)|^p\rangle \propto |\tau|^{\zeta_p^{(d)}}$ should be used to test for intermittency while spectral analysis or correlation analysis should be done on degree d differences, $E_{\delta_\Delta^{(d)}\eta}(\omega) \propto |\omega|^{\beta^{(d)}}$ and $C_{\delta_\Delta^{(d)}\eta}(\theta) \propto |\theta|^{-\kappa^{(d)}}$. The relationships between the scaling exponents then become

$$\zeta_2^{(d)} = n-1, \quad \kappa^{(d)} = 1+2d-n, \quad \beta^{(d)} = n-2d, \qquad (9)$$

for $n < 1+2d$. In other words, the proposed procedure removes the biases in the estimation of the scaling expo-

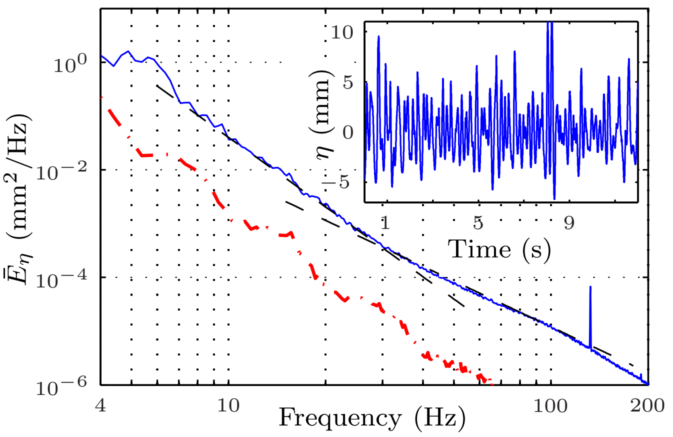

Fig. 1: Power spectra of the experimental data $\eta(t)$ (solid line) and of a synthetic signal (dash-dotted line). Dashed lines have slopes of -4.3 and -2.8. Inset: Typical temporal evolution of $\eta(t)$ during 10 s, $\langle\eta\rangle \simeq 0$.

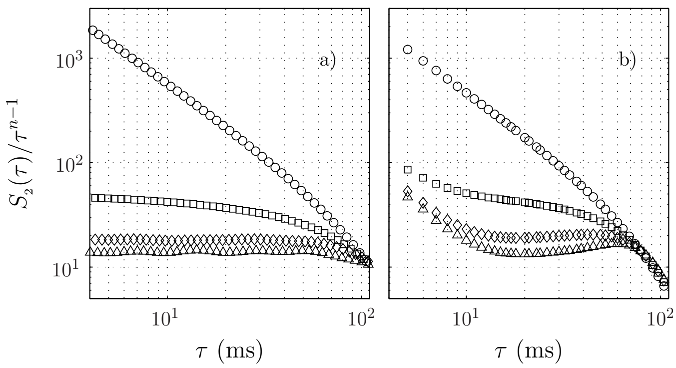

Fig. 2: Rescaled second-order moment of the structure functions $\mathcal{S}_2(\tau)/\tau^{n-1}$ with $n = 4.3$ computed from the (○) first-, (□) second-, (◇) third- and (△) fourth-degree differences of the signal as a function of the time lag τ. a) Synthetic signal. b) Experimental signal. Correlation time is $\tau_c \simeq 63$ ms.

nents. These biases arise from the local regular behaviors of a signal that can occur at different degrees of the signal differentiability. Hence, in order to numerically check that the signal regular components have been adequately removed, it is good practice to check that results remains consistent when increasing degree d to $d+1$. In particular, one should make sure that $\zeta_p^{(d)} \simeq \zeta_p^{(d+1)}$. Note that for discontinuous signals the use of increments or higher-degree differences is unsuitable, a wavelet-based approach is more suited [14].

Applications. – To illustrate the results of the previous section, let us now apply the proposed estimator based on higher-degree difference statistics to signals with steep spectra and probe their possible intermittent nature. A synthetic signal with prescribed intermittency and experimental data of wave turbulence on a fluid surface will be tested below for comparison. To our knowledge, only one study has compared the method of second-degree differences with more complex estimators based on inverse statistics in order to probe intermittency in a simulation of two-dimensional flows [17].

Synthetic data. Here, we apply the above suggested estimator to a synthetic data. Recently, it has been proposed that the scaling properties of experimental velocity (transverse to the mean flow) in fully developed turbulence could be described by log-normal Random Wavelet Cascade (RWC) [18]. RWC generalizes the concept of self-similar cascades leading to multifractal measures ($-1 \leq n \leq 1$) to the construction of scale-invariant signals ($n > 1$) using orthonormal wavelet basis [18]. Instead of redistributing the measure over sub-intervals with multiplicative weights, it allocates the wavelet coefficients in a multiplicative way on the dyadic grid. This method has been implemented to generate multifractal functions from a given deterministic or probalistic multiplicative process. From a mathematical point of view, the convergence of the cascade and the regularity properties of the so-obtained stochastic functions have been discussed in Ref. [19]. Intermittency of RWC is characterized by the theoretical scaling exponents $\zeta_p = c_1 p - c_2 p^2/2$ [19]. Here, we consider a realization of 3×10^6 data points of the RWC process and choose $c_1 = 1.92$ and $c_2 = 0.27$ to reproduce the intermittent properties of experimental data of wave turbulence (see below).

We first compute the usual power spectrum of the RWC signals allowing us to assess that its frequency power law exponent is around $3 < n < 5$. This suggests that unbiased estimates of the scaling exponents will be obtained using second (or higher) degree difference statistics (see Table 1). The unbiased power spectrum of this signal is then computed using the Fourier transform [see Eqs. (6) and (7)] of the second-degree differences and is shown in Fig. 1. It roughly behaves as a steep power-law over one decade in frequency, i.e. $E(\omega) \propto \omega^{-n}$ with $n \simeq 4.3$. The second-order structure functions $S_2^{(d)}$ of this synthetic signal are then computed using the first, second, third and fourth degree difference statistics ($d = 1$, 2, 3 and 4) as shown in Fig. 2a. We observe that, when using the third-degree statistics ($d = 3$), one has $S_2^{(3)}(\tau) \simeq \tau^{\zeta_2^{(3)}}$ with $\zeta_2^{(3)} = 3.33$ that is in good agreement with the theoretical value $\zeta_2 = 2(c_1 - c_2) = 3.3$, the classical relationship $\zeta_2^{(3)} = n - 1$ being thus well satisfied. When $d = 4$, we obtain $\zeta_2^{(4)} = 3.35$ a value that is consistent with the one found with $d = 3$. For $d = 2$, one obtains $\zeta_2^{(2)} = 3.23$, a value that is slightly below the previous one since it begins to be biased by the smoother part of the signal (two times differentiable or more). Finally, using the usual first-degree statistics ($d = 1$) leads to $\zeta_2^{(1)} \simeq 2$ since the RWC signal is differentiable with a steep power spectrum of exponent $n \geq 3$ (see Table 1).

The estimations of the structure-function exponents $\zeta_p^{(d)}$ versus p using the first, second, third and fourth degree difference statistics ($d = 1$, 2, 3 and 4) are presented in

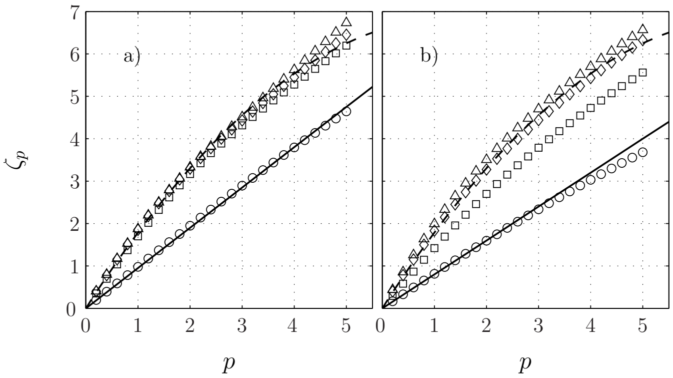

Fig. 3: Structure function exponents, ζ_p, as a function of p for the synthetic signal (a) and experimental data (b). ($\circ$) $\zeta_p^{(1)}$ computed from the first-degree increments ; ($\square$) $\zeta_p^{(2)}$ computed from the second-degree differences and fitted by ($--$) $\zeta_p^{(2)} = c_1 p - c_2 p^2/2$ with $c_1 = 1.92$ and $c_2 = 0.27$; ($\diamond$) $\zeta_p^{(3)}$ and ($\triangle$) $\zeta_p^{(4)}$ computed from the third-degree and fourth-degree differences. Solid lines are linear fits of $\zeta_p^{(1)}$: (a) $\zeta_p^{(1)} = 0.95p$, and (b) $\zeta_p^{(1)} = 0.8p$.

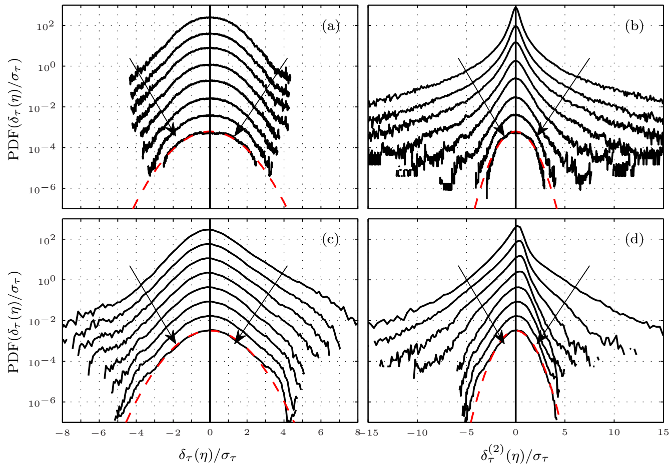

Fig. 4: PDFs of the first-degree increments $\delta_\tau^{(1)}\eta$ for different τ from 6 to 100 ms (see arrows): (a) Synthetic data, and (c) Experimental data. PDFs of the second-degree differences $\delta_\tau^{(2)}\eta$ for $6 \leq \tau \leq 100$ ms: (b) Synthetic data, and (d) Experimental data. Dashed-line: Gaussian with zero mean and unit standard deviation. σ_τ are the rms values. Each curve has been shifted vertically for clarity.

Fig. 3a. We observe that $\zeta_p^{(2)}$ is a non-linear function of p which provides a clear evidence of the intermittent nature of the RWC signal fluctuations. The fact that $\zeta_p^{(3)}$ and $\zeta_p^{(4)}$ estimates are consistent with the previous ones ($\zeta_p^{(2)} \simeq \zeta_p^{(3)} \simeq \zeta_p^{(4)}$) provides further confidence on this diagnosis. It is noteworthy that $\zeta_p^{(2)}$, $\zeta_p^{(3)}$ and $\zeta_p^{(4)}$ estimates are in good agreement with the theoretical expectation $\zeta_p = c_1 p - c_2 p^2/2$, which illustrates that the proposed framework allows an accurate characterization of the intermittent properties. Finally, as expected for a differentiable signal, using first-degree increments leads to $\zeta_p^{(1)}$ is a linear function of p with a slope close to 1. This latter result thus leads to a misleading conclusion that the RWC signal does not present intermittency. This exemplifies the need to adapt the degree of the difference statistics to the steepness of the power spectrum.

Finally, the probability density functions (PDFs) of the first-degree increments of the synthetic signal are plotted in Fig. 4a for different time lags τ. All the PDFs have the same shape independent of the scale τ, thus showing no intermittency. In contrast, the PDFs of the second-degree increments displayed in Fig. 4b show a clear evolution across scales, highlighting the intermittency of the signal. This is consistent with the previous structure function analysis, and further underlines that high-degree difference statistics is needed to test intermittency of steep power-spectrum signals.

Wave turbulence data. We now apply the above proposed statistical estimator on an experimental signal of hydrodynamics surface wave turbulence [20]. A typical signal is the temporal evolution of the surface wave amplitude, $\eta(t)$, measured at a given location of the free surface of the fluid (see inset of Fig. 1). Data are recorded from 10 successive experiments of 300 s each where surface waves are generated by a wave maker driven by random noise forcing in a frequency range 0–6 Hz [20]. Wave heights was measured at 1 kHz sampling rate ($\Delta = 1$ ms) resulting in 3×10^6 data points. As for the RWC signal, the initial estimation of the power spectrum steepness indicates that $3 < n < 5$. In order to probe for possible intermitent properties of this signal with such a steep power spectrum, we thus need to use the adapted difference statistics proposed in the previous section.

The power spectrum of $\eta(t)$, estimated using the Fourier transform [see Eqs. (6) and (7)] of $\delta_\Delta^{(2)}\eta$, is shown in Fig. 1. It displays two frequency ranges with a power-law behaviour. In the low-frequency spectrum range ($\sim 7-30$ Hz) corresponding to the gravity wave turbulence regime, we observe $E(\omega) \propto \omega^{-4.3}$ while the high-frequency range ($\sim 30-100$ Hz) corresponding to the capillary regime is characterized by $E(\omega) \propto \omega^{-2.8}$ [7]. The second-order structure functions $S_2^{(d)}(\tau)$ of $\eta(t)$ for $d = 1$, 2, 3 and 4 are shown in Fig. 2b as a function of the time lag $5 \leq \tau \leq 100$ ms. When considering structure functions $S_2^{(d)}$ for $d \geq 2$ in the time lag range $\tau \lesssim 80$ ms (corresponding to frequencies above the maximal forcing frequency 6 Hz), we roughly observe two different power-law scaling behaviours in the gravity regime $15 \lesssim \tau \lesssim 65$ ms and the capillary regime $5 \lesssim \tau \lesssim 15$ ms [a time lag τ corresponds to a frequency $f = 1/(2\tau)$]. We will focus below only to the gravity regime since the transition between both regimes in Fig. 2b occurs rather smoothly which significantly reduce the time lag range available to fit the scaling exponent in the capillary regime. As explained above, for $d = 1$, $S_2^{(1)}$ is dominated by the signal differentiability and does not display both scaling regimes. Focusing only on the gravity regime ($15 \lesssim \tau \lesssim 65$ms), one consistently finds $\zeta_2^{(3)} = 3.3$ and $\zeta_2^{(4)} = 3.4$ in good agreement with the spectral exponent $n = 4.3$. When using $d = 2$, the smoother transition observed between the scaling regimes leads to a slightly underestimated value $\zeta_2^{(2)} = 2.9 \neq n - 1$.

We then look for possible intermittent properties of the turbulence wave data in the gravity regime. The evolution of $\zeta_p^{(d)}$ with p is shown in Fig. 3b for $d = 1$, 2, 3 and 4. When using the first-degree increments, $\zeta_p^{(1)} \simeq 0.8p$ is a linear function of p. As underlined above, in presence of a steep power spectrum, $\zeta_p^{(1)}$ is dominated by the differential component of the signal masking possible intermittency. For $d = 2$, 3 and 4, we observe a clear non linear behaviour of $\zeta_p^{(d)}$ versus p. We note that while $\zeta_p^{(3)}$ and $\zeta_p^{(4)}$ provide consistent estimates of ζ_p for all p, $\zeta_p^{(2)}$ estimates are slightly below these latters as already observed above for $p = 2$. Finally, the coherence between the estimates of ζ_p for two successive values of the difference degree ($\zeta_p^{(3)} \simeq \zeta_p^{(4)}$) and with the spectral analysis ($\zeta_2^{(3,\,4)} \simeq n - 1$) strongly suggests that these measurements are reliable. We can thus conclude these data of wave turbulence are intermittent. Fitting of $\zeta_p^{(3)}$ with the polynomial model $\zeta_p = c_1 p - c_2 p^2/2$ yields $c_1 = 1.9$ and an intermittency coefficient $c_2 = 0.27$.

Another way to highlight the wave turbulence intermittency is to observe a shape deformation of the probability density functions (PDFs) of the signal increments with the time lag τ. The PDFs of the first and second-degree increments of the wave amplitudes are respectively plotted in Figs. 4c and 4d for $6 \leq \tau \leq 100$ ms. When τ is increased, the PDF's shape of the second-degree increments changes continuously up to a nearly Gaussian shape at large τ (see Fig. 4d). This deformation is a direct signature of intermittency. As predicted above, this intermittency is not diagnose when using the first-degree estimator: almost no deformation of the PDF shapes of the first-degree increments is observed in Fig. 4c.

In this Letter, we have proposed an easily applicable framework based on high-degree difference statistics to probe for possible intermittency of a signal with a steep power spectrum. We applied it to synthetic data and to wave turbulence data. This has led to the observation of wave turbulence intermittency [20]. In the same way, it can be used on previous existing data notably of magneto-hydrodynamic [4] or two-dimensional turbulence [8], both showing steep power spectra.

* * *

We thank A. Newell, F. Pétrélis, and A. Arneodo for fruitfull discussions. This work has been supported by ANR Turbonde BLAN07-3-198746.

REFERENCES

[1] Kolmogorov A. N., *Dokl. Akad. Nauk SSSR*, **30** (1941) 9; **32** (1941) 16, reproduced in *Proc. R. Soc. Lond. A* **434** (1991) 9; **434** (1991) 15

[2] Pope S. B., *Turbulent Flows* (Cambridge University Press) 2000

[3] Frisch U., *Turbulence* (Cambridge University Press) 1995. This approach was first introcuced in Parisi G. and Frisch U., *Proc. Intern. School on Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam) 1985, p. 84.

[4] Bourgoin M. et al., *Phys. Fluids*, **14** (2002) 3046.

[5] Julian R., Whasington W. M., Hembree, L. and Ridley C., *J. Atmos. Sci.*, **27** (1970) 376; Morel P. and Larcheveque M., *J. Atmos. Sci.*, **31** (1974) 2189; Desbois M., *J. Atmos. Sci.*, **32** (1975) 1838

[6] Toba Y., *J. Oceanogr. Soc. Jpn.*, **29** (1973) 209; K. K. Kahma, *J. Phys. Oceanogr.* **11**, 1503 (1981); G. Z. Forristall, *J. Geophys. Res. Oceans Atmos.* **86**, 8075 (1981); M. A. Donelan *et al.*, *Philos. Trans. R. Soc. A* **315**, 509 (1985)

[7] Falcon E., Laroche C. and Fauve S., *Phys. Rev. Lett.*, **98** (2007) 094503, and references therein.

[8] Paret J., Jullien M.-C. and Tabeling P., *Phys. Rev. Lett.*, **83** (1999) 3418; Tabeling P., *Phys. Reports*, **362** (2002) 1.

[9] Rutgers M. A., *Phys. Rev. Lett.*, **81** (1998) 2244, and references therein.

[10] Babiano A., Basdevant C. and Sadourny R., *C. R. Acad. Sc. Paris*, **299** (1984) 495 (in french); *J. Atmos. Sci.*, **42** (1985) 941 and references therein

[11] Davidson P. A. and Pearson B. R., *Phys. Rev. Lett.*, **95** (2005) 214501

[12] Monin A. S. and Yaglom A. M., *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. **2** (The MIT Press) 1975

[13] Kellay H., Wu X-l. and Goldburg W. I., *Phys. Rev. Lett.*, **74** (1995) 20; Riviera M., Vorobieff P. and Ecke R. E., *Phys. Rev. Lett.*, **81** (1998) 1417

[14] Muzy J. F., Bacry E. and Arneodo A., *Phys. Rev. Lett.*, **67** (1991) 3515; *Phys. Rev. E*, **47** (1993) 875; *Int. J. Bifurc. Chaos*, **4** (1994) 245-302

[15] Jensen M. H., *Phys. Rev. Lett.*, **83** (1999) 76

[16] Samorodnisky G. and Taqqu M. S, *Stable Non-Gaussian Random Processes* (Chapman and Hall, New York) 1994.

[17] Biferale L., Cencini M., Lanotte A. S. and Vergni D., *Phys. Fluids*, **15** (2003) 1012

[18] Arneodo A., Muzy J. F. and Roux S.G., *J. Phys. II France*, **7** (1997) 363

[19] Arneodo A., Bacry E. and Muzy J. F., *J. Math. Phys.*, **39** (1998) 4142

[20] Falcon E., Fauve S. and Laroche C., *Phys. Rev. Lett.*, **98** (2007) 154501